NEW APPLICATIONS OF MACLAURIN'S SERIES IN THE SOLUTION OF EQUATIONS AND IN THE EXPANSION OF FUNCTIONS.

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I.—INTRODUCTION.

The modern theory of differential equations is based on the expansion by Maclaurin's series of the solutions of the equations in infinite series. The striking analogy existing between the theory of algebraic equations and the theory of differential equations suggested the possibility of expressing the solutions of algebraic equations in series to be obtained by an application of Maclaurin's series. After some experimenting the author happened on the device of introducing a factor x into all the terms but two of the equation $f(y)=0$, whereby y becomes an implicit function of x. The successive x-derivatives of y are now formed, and together with y are evaluated for $x=0$. By Maclaurin's series the expansions of y in powers of x become known. If x be made unity in these expansions, the roots of $f(y)=0$ are found, provided the resulting series are convergent.

To illustrate this method, consider the equation

$$(1)\quad y^4 - 3y^2 + 75y - 10000 = 0.$$

Maclaurin's series

$$y = y_0 + \frac{dy_0}{dx_0}x + \frac{d^2y_0}{dx_0^2}\frac{x^2}{2!} + \frac{d^3y_0}{dx_0^3}\frac{x^3}{3!} + \frac{d^4y_0}{dx_0^4}\frac{x^4}{4!} + \cdots,$$

where $y_0, \frac{dy_0}{dx_0}, \frac{d^2y_0}{dx_0^2}, \frac{dy^3_0}{dx_0^3}, \frac{d^4y_0}{dx_0^4}, \ldots$ stand for the values of $y, \frac{dy}{dx}, \frac{d^2y}{dx^2}, \frac{d^3y}{dx^3}, \frac{d^4y}{dx^4}, \ldots$ when x is made zero, expands y, a function of x, in powers of x.

By introducing a factor x in the second and third terms of (1) an equation is formed

$$(2)\quad y^4 - 3xy^2 + 75xy - 10000 = 0$$

which defines y as an implicit function of x.

Differentiating (2) twice in succession

$$(3)\quad 4y^3\frac{dy}{dx} - 3y^2 + 75y - 6xy\frac{dy}{dx} + 75x\frac{dy}{dx} = 0$$

$$(4)\quad 4y^3\frac{d^2y}{dy^2} + 12y^2\left(\frac{dy}{dx}\right)^2 - 12y\frac{dy}{dx} + 150\frac{dy}{dx} - 6x\left(\frac{dy}{dx}\right)^2 -$$

$$6xy\frac{d^2y}{dx^2} + 75x\frac{d^2y}{dx^2} = 0.$$

Making x zero in (2), (3) and (4)

$$y_0 = + 10, - 10, + 10\sqrt{-1}, - 10\sqrt{-1}$$

$$\frac{dy_0}{dx_0} = -.1125, -.2625, +.1875 - .075\sqrt{-1}, +.1875 + .075\sqrt{-1}$$

$$\frac{d^2y_0}{dx_0^2} = -.0029, -.0029, -.0000015 + .0039\sqrt{-1}, -.0000015 - .0039\sqrt{-1}.$$

Substituting these four sets of values in Maclaurin's series and placing $x = 1$, the roots of equation (1) are found to be

$$y_1 = +9.886,\ y_2 = -10.261,\ y_3 = +.1875 + 9.927\sqrt{-1},$$
$$y_4 = +.1875 - 9.927\sqrt{-1},$$

all correct to the last decimal.

This method will be applied to the solution (II) of trinomial algebraic equations, (III) of general algebraic equations, (IV) of trinomial transcendental equations, and finally (V) the method will be applied to obtain expansions commonly obtained by Lagrange's series.

II.—Trinomial Algebraic Equations.

The general trinomial equation of degree n has the form

$$(1)\quad y^n - nay^{n-k} - b = 0.$$

Introducing a factor x in the second term of (1)

$$(2)\quad y^n - naxy^{n-k} - b = 0.$$

Applying the method and denoting the n^{th} root of b by ω

$$(3)\quad y = \omega + \omega^{1-k}a + \omega^{1-2k}(1 - 2k + n)\frac{a^2}{2\,!}$$

$$+ \omega^{1-3k}(1 - 3k + n)(1 - 3k + 2n)\frac{a^3}{3\,!}$$

$$+ \omega^{1-4k}(1 - 4k + n)(1 - 4k + 2n)(1 - 4k + 3n)\frac{a^4}{4\,!} + \ldots .$$

To determine when series (3) is convergent, group the terms numbered 1, $n+1$, $2n+1$, $3n+1$, ... , then those numbered 2, $n+2$, $2n+2$, $3n+2$, ... , finally those numbered n, $2n$, $3n$, $4n$, ... Each of these n partial series is found by Cauchy's ratio test to be convergent when a^n is numerically less than $k^{-k}(n-k)^{k-n}b^k$. When this condition of convergency is satisfied series (3), by substituting for ω in succession each of the n values of the n^{th} root of b, determines the n roots of equation (1).

By introducing the factor x in the third term of equation (1) and applying the method a series is obtained which determines k roots of equation (1), and by introducing the factor x in the first term of equation (1) a series is obtained which determines $n-k$ roots of equation (1). The two series thus obtained are convergent when a^n is numerically greater than $k^{-k}(n-k)^{k-n}b^k$. When $a^n=k^{-k}(n-k)^{k-n}b^k$ equation (1) has equal roots. There is therefore developed a complete theory of trinomial equations.

The general fifth degree equation can, by Tschirnhausen transformations requiring the solution of equations of the second and third degrees only, be transformed into the trinomial equation $y^5+ay+b=0$. If a^5 is numerically less than $\frac{625}{256}b^4$, the five roots of this equation are found by applying the method to $y^5+axy+b=0$. If a^5 is numerically greater than $\frac{625}{256}b^4$, the five roots are found by applying the method to $y^5+ay+bx=0$ and $xy^5+ay+b=0$. If a^5 numerically equals $\frac{625}{256}b^4$, the fifth degree equation has equal roots, and the removal of the equal roots makes the solution of the fifth degree equation depend on the solution of an equation of a degree not higher than the third. A third degree equation becomes trinomial by removing the second term, which is accomplished by a linear transformation. The method of this paper therefore effects the complete solution of the general fifth degree equation in infinite series.

In Weber's Algebra, volume I, pages 396–399, the real and imaginary roots of the equation $x^3-2x-2=0$ are computed by a method invented by Gauss for the solution of trinomial equations. The convergency test shows that the series found by introducing the variable factor in the second term is convergent. Now the mathematician is satisfied when the convergency of the infinite series he uses is established, but the computer desires that the infinite series he is obliged to use shall converge rapidly. By transforming the equation $x^3 - 2x - 2 = 0$ into another lacking the first power, which is accomplished by placing $x = \frac{3}{3y-1}$, the equation $54y^3 - 18y - 23 = 0$ is found. The series found by applying the method to $54y^3 - 18xy - 23 = 0$ converges much more rapidly than the series obtained from the original equation. Differentiating $54y^3 - 18xy - 23 = 0$ four times in succession and making x zero,

$$y_0 = \quad .7524, \; -.3762 \pm .3762\sqrt{-3}$$

$$\frac{dy_0}{dx_0} = \quad .1477, \; -.0738 \mp .0738\sqrt{-3}$$

$$\frac{1}{2!}\frac{d^2y_0}{dx_0^2} = \quad 0, \quad 0$$

$$\frac{1}{3!}\frac{d^3y_0}{dx_0^3} = -.0019, \; +.0010 \mp .0010\sqrt{-3}$$

$$\frac{1}{4!}\frac{d^4y_0}{dx_0^4} = \quad .0004, \; -.0002 \mp .0002\sqrt{-3}.$$

The three values of y are .889 and $-.4492 \pm .3012\sqrt{-3}$, the corresponding values of x are 1.768 and $-.8847 \mp .5898\sqrt{-1}$. If the computations are made by logarithms they are not very lengthy.

The equation $y^4 - 11727\,y + 40385 = 0$ occurs in a paper by Mr. G. H. Darwin "On the Precession of a Viscous Spheroid," published in the *Philosophical Transactions of the Royal Society*, Part II, 1879, page 508. The convergency test shows that the factor x must be introduced in the last and in the first terms. The equation therefore has two real positive and two imaginary roots. Applying the method to

$$y^4 - 11727y + 40385x = 0,$$

$$y_0 = \quad 22.720, \ -11.360 \pm 11.360\sqrt{-3}$$

$$\frac{dy_0}{dx^0} = - \ 1.148, \ - \ 1.148$$

$$\frac{1}{2!}\frac{d^2y_0}{dx_0^2} = - \ .116, \qquad .058 \pm \ .058\sqrt{-3}$$

$$\frac{1}{3!}\frac{d^3y_0}{dx_0^3} = - \ .019, \qquad .010 \mp \ .010\sqrt{-3}$$

$$\frac{1}{4!}\frac{d^4y_0}{dx_0^4} = - \ .004, \ - \ .004$$

Three roots of the equation are 21.432 and $12.444 \pm 19.759\sqrt{-1}$. Applying the method to

$$xy^4 - 11727y + 40385 = 0,$$

$$y_0 = 3.4436, \ \frac{dy_0}{dx_0} = .0120, \ \tfrac{1}{2}\ \frac{d^2y_0}{dx_0} = .0002.$$

The fourth root of the equation is 3.4558.

This method applied to trinomial equations proves that an equation of degree n has n roots, determines how many roots are real, and presents a uniform scheme for computing all the roots, real and imaginary.

III.—General Algebraic Equations.

The method applied to the complete equation of degree n furnishes $\frac{n\,(n-1)}{1.2}$ series, and it becomes necessary to determine which of these series give n convergent series for the roots of the equation and if possible to insure rapidity of convergence of these n series.

Suppose the equation of degree n to be

$$\underline{ay^{n}} + a_0y^{n-1} + \ldots + a_k\,y^{n-k+1} + \underline{by^{n-k}} + b_0y^{n-k-1} + \ldots$$
$$+ b_1y^{n-k-l+1} + \underline{cy^{n-k-l}} + c_0y^{n-k-l-1} + \ldots + c_my^{n-k-l-m+1}$$
$$+ \underline{dy^{n-k-l-m}} + d_0y^{n-k-l-m-1} + \ldots + ry + \underline{s} = 0,$$

and suppose the terms which are underscored to be the terms from which the two terms into which the factor x is not introduced must be selected by taking consecutive terms in regular order from the left. The problem is how to recognize the terms which must be underscored.

If the factor x is omitted from the first two underscored terms $y_0=\left(-\frac{b}{a}\right)^{\frac{1}{k}}$; if from the second and third underscored terms $y_0=\left(-\frac{c}{b}\right)^{\frac{1}{l}}$; if from the third and fourth underscored terms $y_0=\left(-\frac{d}{c}\right)^{\frac{1}{m}}$; if from the last two underscored terms $y_0=\left(-\frac{s}{d}\right)^{\frac{1}{n-k-l-m}}$. Altogether n values of y_0 are found, and it is seen at a glance what values of y_0 are real and what are imaginary. In order that these values of y_0 shall be close approximations of the roots of the given equation, the successive derivatives $\frac{dy_0}{dx_0}$, $\frac{d^2y_0}{dx_0^2}$, $\frac{d^3y_0}{dx_0^3}$, $\frac{d^4y_0}{dx_0^4}$, ... must be small. Forming $\frac{dy_0}{dx_0}$ corresponding to $y_0=\left(-\frac{b}{a}\right)^{\frac{1}{k}}$ and assuming that c is of such a magnitude that the term containing c overshadows all the other terms in the numerator of $\frac{dy_0}{dx_0}$, it is found that $\frac{dy_0}{dx_0}$ is necessarily small if the ratio of b^{k+l} to a^lc^k is numerically large. This same condition insures that the following derivatives $\frac{d^2y_0}{dx_0^2}$, $\frac{d^3y_0}{dx_0^3}$, ... are small.

In like manner it is shown that the derivatives corresponding to $y_0=\left(-\frac{c}{b}\right)^{\frac{1}{l}}$ are small provided the ratio of c^{l+m} to b^md^l is numerically large, and that the derivatives corresponding to $y_0=\left(-\frac{d}{c}\right)^{\frac{1}{m}}$ are small provided the ratio of d^{n-k-l} to $c^ms^{n-k-l-m}$ is numerically large. This ratio should, if possible, be made larger than 10 to insure rapid convergence.

The directions for underscoring terms are therefore as follows:

Underscore the first and last terms of the equation. Such other terms are to be underscored as satisfy the condition that if any three consecutive underscored terms be chosen, the ratio of the coefficient of the middle term with an exponent equal to the difference of the degrees of the first and third terms to the product of the coefficient of the first of the three terms with an exponent equal to the difference of the degrees of the second and third terms and the coefficient of the third term with an exponent equal to the difference of the degrees of the first and second terms shall be a large number.

To illustrate the method, the following equations are discussed :

$$(a)\quad \underline{y^5} - \underline{10y^3} + \underline{6y} + \underline{1} = 0.$$

Here all the terms are underscored, for the ratio of 10^4 to 6^2 is large, and the ratio of 6^3 to 10 is large. The method must be applied to (1) $y^5 - 10y^2 + 6xy + x = 0$, (2) $xy^5 - 10y^3 + 6y + x = 0$ and (3) $xy^5 - 10xy^3 + 6y + 1 = 0$. The computation determines the following values:

From (1)
$$y_0 = +3.167, \quad -3.167$$
$$\frac{dy_0}{dx_0} = -0.100, \quad +0.090$$
$$\tfrac{1}{2}\frac{d^2y_0}{dx_0^2} = -0.008, \quad +0.008;$$

From (2)
$$y_0 = +0.775, \quad -0.775$$
$$\frac{dy_0}{dx_0} = +0.107, \quad +0.060$$
$$\tfrac{1}{2}\frac{d^2y_0}{dx_0^2} = -0.006, \quad +0.016;$$

From (3)
$$y_0 = +0.166, \quad \frac{dy_0}{dx_0} = -0.007.$$

The roots of the given equation are $y_1 = +3.05$, $y_2 = -3.06$, $y_3 = +0.87$, $y_4 = -0.69$, $y_5 = -0.17$.

$$(b)\quad x^4 + 4x^3 - 4x^2 - 11x + 4 = 0.$$

Here the terms to be underscored in addition to the first and last are probably the second and fourth, but as the ratio of 4^3 to 11 is rather small, it is safer to transform the equation into another lacking the second term by the substitution $x = y - 1$. There results

$$\underline{y^4} - \underline{10y^2} + 5y + \underline{8} = 0.$$

The terms to be underscored are the first, second and last and the roots are obtained by applying the method to

$y^4 - 10y^2 + 5xy + 8x = 0$ and $xy^4 - 10y^2 + 5xy + 8 = 0$. From each of the two equations two real roots, one positive and one negative, are found.

$$(c)\quad 7x^4 + 20x^3 + 3x^2 - 16x - 8 = 0.$$

Here the terms to be underscored are probably the first, second and last, indicating the existence of two imaginary and two real roots, one positive and one negative. All doubt is removed by transforming by $x = y - .7$ into

$$\underline{7y^4} + .4y^3 - \underline{18.42y^2} - 1.404y - \underline{.5093} = 0.$$

The transformation $x = y - .7$ is selected because it is a simple transformation which makes the coefficient of the second term very small.

$$(d)\quad x^5 + 12x^4 + 59x^3 + 150x^2 + 201x - 207 = 0.$$

Here probably only the first and last terms are to be underscored, indicating the existence of four imaginary roots and one real positive root. Transforming by $x = y - 2$, which makes the coefficient of the second term small,

$$\underline{y^5} + 2y^4 + 3y^3 + 4y^2 + 5y - \underline{321} = 0.$$

The roots are found by applying the method to

$$y^5 + 2xy^4 + 3xy^3 + 4xy^2 + 5xy - 321 = 0$$

$$(e)\quad x^4 - 80x^3 + 1998x^2 - 14937x + 5000 = 0.$$

Here probably every term should be underscored, indicating four positive real roots. Transforming by the substitution $x = y + 20$,

$$y^4 - 402y^2 + 983y + 25460 = 0.$$

Here the terms to be underscored are the first, second and last. More rapidly convergent series are found by reversing the last equation,

$$25460v^4 + 983v^2 - 402v + 1 = 0,$$

and making the substitution $v = z - .01$, whence

$$\underline{25460z^4} - 35.4z^3 - \underline{416.214z^2} + 8.23306z + \underline{.9590716} = 0.$$

When z has been computed, x is found from

$$x = \frac{2000z + 80}{100z - 1}.$$

Only linear transformations which make the coefficient of the second term of the complete equation or of the equation reversed zero or small are used, as other transformations become too complicated to make the method practicable.

IV.—Transcendental Trinomial Equations.

Let an equation of the form $y+af(y)+b=0$, where $f(y)$ is a transcendental function, be called a transcendental trinomial equation. Such equations are readily solved by the method, provided the resulting series is rapidly convergent, but in the absence of a transformation which insures rapid convergence the method has little practical value.

Suppose the equation $2y+\log y - 1000=0$ to be given. Applying the method to $2y+x\log y - 1000=0$, if the Napierian logarithm of y is taken, $y_0=5000$, $\frac{dy_0}{dx_0} = -4.30625$, $\frac{1}{2}\frac{d^2y_0}{dx_0^2} = +0.000215$, and $y=4995.69$; if the common logarithm of y is taken, $y_0=5000$, $\frac{dy_0}{dx_0} = -1.84948$, $\frac{1}{2}\frac{d^2y_0}{dx_0^2} = +0.00018$, and $y=4998.15$.

V.—Expansions.

If $y=z+v\varphi(y)$, where v and z are independent variables, Lagrange's series expands any function of y in powers of v. These expansions may be obtained by writing $y=z+vx\varphi(y)$ and expanding $f(y)$, which now becomes a function of x, by Maclaurin's series and making x unity in the result.

The method will be illustrated by obtaining two expansions which occur in theoretical astronomy. From the equation $E=M+e \sin E$, where E is the eccentric anomaly, M the mean anomaly and e the eccentricity of the orbit, it is necessary to find E and $(1 - e \cos E)^{-2}$.

To find E, write $E=M+ex \sin E$, whereby E becomes an implicit function of x. Differentiating twice in succession with respect to x,

$$\frac{dE}{dx} = e \sin E + ex \cos E \frac{dE}{dx},$$

$$\frac{d^2E}{dx^2} = 2\, e \cos E \frac{dE}{dx} + ex \cos E \frac{d^2E}{dx^2} - ex \sin E \left(\frac{dE}{dx}\right)^2.$$

Making x zero, $E_0 = M$, $\frac{dE_0}{dx_0} = e \sin M$, $\frac{d^2E_0}{dx_0^2} = 2e^2 \cos M \sin M$.

Substituting in Maclaurin's series and making x unity,

$$E = M + e \sin M + \frac{e^2}{2} \sin (2M) + \ldots .$$

To find $(1 - e\cos E)^{-2}$, write $E = M + ex\sin E$ and $y = (1 - e \cos E)^{-2}$. Since y is a function of x through E,

$$\frac{dy}{dx} = -2e\,(1 - e \cos E)^{-3} \sin E \frac{dE}{dx}$$

$$\frac{d^2y}{dx^2} = 6e^2\,(1 - e \cos E)^{-4} \sin^2 E \frac{dE}{dx}$$

$$- 2e\,(1 - e \cos E)^{-3} \cos E \left(\frac{dE}{dx}\right)^2$$

$$- 2e\,(1 - e \cos E)^{-3} \sin E \frac{d^2E}{dx^2}$$

Placing $x = 0$, when $E = M$, $\frac{dE}{dx} = e \sin M$ and

$\frac{d^2E}{dx^2} = 2e^2 \sin M \cos M$,

$$y_0 = (1 - e \cos M)^{-2},$$

$$\frac{dy_0}{dx_0} = -2e^2\,(1 - e \cos M)^{-3} \sin^2 M,$$

$$\frac{d^2y_0}{dx_0^2} = 6e^4\,(1 - e \cos M)^{-4} \sin^4 M,$$

$$- 6e^3\,(1 - e \cos M)^{-3} \sin^2 M \cos M.$$

Substituting in Maclaurin's series and making x unity,

$$(1 - e \cos E)^{-2} = (1 - e \cos M)^{-2} - 2e^2\,(1 - e \cos M)^{-3} \sin^2 M$$

$$+ 3e^4\,(1 - e \cos M)^{-4} \sin^4 M$$

$$- 3e^3\,(1 - e \cos M)^{-3} \sin^2 M \cos M + \ldots .$$

In like manner all expansions obtained by Lagrange's series may be obtained by a direct application of Maclaurin's series. Of course it is evident that if e is considered a variable the derivatives with respect to e may be formed and the introduction of x is unnecessary.

Historical Note.

Lagrange, in the memoir "Nouvelle methode pour resoudre les Equations Litterales par le moyen des Series," read before the Berlin Academy in 1770, found all the roots of an equation in infinite series. McClintock, in Volume xvii of the *American Journal of Mathematics*, obtained by his Calculus of Enlargement series better adapted to computation. It was recognized that these series may be obtained by Lagrange's series. McClintock calls the coefficients of the terms which have been underscored the dominants of the equation. The method of the present paper brings the computation of the roots of equations by means of series within the range of elementary instruction.

Since completing this paper the author found in an extract of a letter from Cauchy to Coriolis, of January 29, 1837, published in the *Comptes Rendus* of the Paris Academy, an announcement of important results to be obtained by breaking up an equation into two parts and introducing as a factor a parameter into one part, which parameter is ultimately to be made unity. In a postscript Cauchy states he discovered the advantage of making one part a binomial. But the author has been unable to find the method sketched in this letter developed. It would indeed be surprising if a method so strikingly direct had escaped notice.

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